Test 2 Problem 3

$f(x,y) = -x^2 + y^2 + 2x$ constrained by $x^2 + y^2 = 4$

If we let $g(x,y) = x^2 + y^2$

then we look for the points where the contour lines of each function are tangent. At these points, the gradients of each function must be parallel. therefore

$$\nabla f = \lambda \nabla g$$

$$\begin{cases} \nabla f = (-2x+2)\vec{i} + 2y\vec{j} \\ \nabla g = 2x\vec{i} + 2y\vec{j} \end{cases}$$

System of three equations in three variables

$$\begin{cases} \Rightarrow -2x+2 = \lambda 2x \\ \text{and} \quad 2y = \lambda 2y \\ \text{also} \quad x^2+y^2 = 4 \end{cases}$$

← this keeps us on the contour $x^2+y^2=4$

Solving the first two equations for λ:

$$\lambda = \frac{-2x+2}{2x} = \frac{-x+1}{x}$$

and $\lambda = \frac{2y}{2y} = 1$ if $y \neq 0$

therefore $\frac{-x+1}{x} = 1 \Rightarrow \boxed{x = \frac{1}{2}}$

Because we want all possible solutions to our system, we need to consider the option that $y=0$ from our second equation that is,

the equation $2y = \lambda 2y$ can be true if $\boxed{y=0}$

From our third equation, $x^2+y^2=4$, we have

$$x = \frac{1}{2} \Rightarrow y = \pm \frac{\sqrt{15}}{2}$$

and

$y = 0 \Rightarrow x = \pm 2$

This gives us 4 points that our max and min will come from.

$(2, 0) \qquad f(2,0) = 0$

$(-2, 0) \qquad f(-2,0) = -8 \leftarrow$ minimum f

$\left(\frac{1}{2}, \frac{\sqrt{15}}{2}\right) \qquad f\left(\frac{1}{2}, -\frac{\sqrt{15}}{2}\right) = \frac{9}{2}$

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$\left.\right\}$ maximum f

If you are curious about solutions to our system of equations that show λ also, here they are:

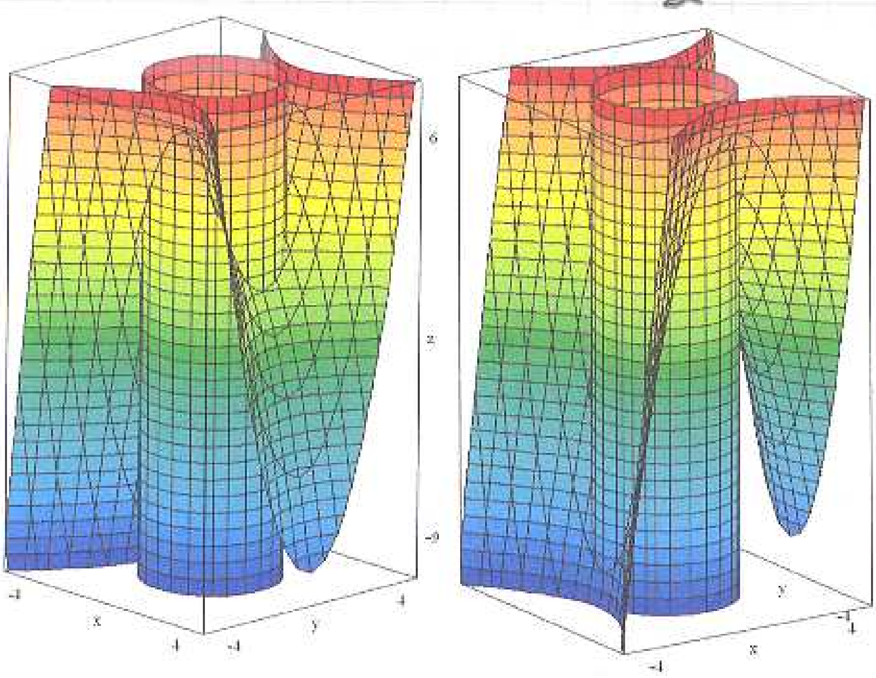
$x = 2,\ y = 0,\ \lambda = -\frac{1}{2}$

$x = -2,\ y = 0,\ \lambda = -\frac{3}{2}$

$x = \frac{1}{2},\ y = \frac{\sqrt{15}}{2},\ \lambda = 1$

$x = \frac{1}{2},\ y = -\frac{\sqrt{15}}{2},\ \lambda = 1$

Two views of the graph



Contour Lines